physics, can give rise to interesting nonlocal effects even observable experimentally, and this is due to the geometrical nature of GR and has nothing to do with the particular form of the field equations of gravitation considered.

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[1] T. M. Helliwell and D. A. Konkowski "Cosmic strings: Gravitation without curvature," Am. J. Phys. **55**, 401–407 (1987).

[2] G. Francisco and G. Matsas, "A remark on the gravitational field produced by an infinite string," Am. J. Phys. **57**, 359–361 (1989).

[3] There are several models for the detailed structure of CSs. Here, I only consider the simplest one, in which the CS is deprived of structure, as used, for instance, in J. R. Gott III, "Gravitational lensing effect of vacuum strings: Exact solutions," Astrophys. J. **288**, 422–427 (1985). Other models, where the CS is obtained as a global exact solution of the Einstein–Yang–Mills–Higgs equations have a more complicated stress energy tensor as analyzed in D. Garfinkle, "General relativistic strings," Phys. Rev. D **32**, 1323–1329 (1985). The relation between the two models is at present not completely understood, as discussed in T. Futamase and D. Garfinkle, "What is the relation between $\Delta\phi$ and μ for a cosmic string?" Phys. Rev. D **37**, 2086–2091 (1988). However, the stress tensor of the global models still has vanishing gravitational mass at least in the linear approximation; see D. Harari and P. Sikivie, "Gravitational field of a global string," Phys. Rev. D **37**, 3438–3440 (1988).

[4] For most types of "everyday" objects, however (except for photons where $p=\rho/3$), the pressure in natural units is totally negligible compared to the energy density. In the case of the CS not only is it not negligible, but it is exactly equal in absolute value.

[5] P. J. Peebles, *Physical Cosmology* (Princeton U. P., Princeton, NJ, 1971); M. Ryan and L. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton U. P., Princeton, NJ, 1975).

[6] A Vilenkin, "Gravitational interactions of cosmic strings," in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge U. P., New York, 1987), pp. 499–523.

[7] The whole issue of existence of CS in GR has been questioned by important conceptual reasons by R. Geroch and J. Traschen, Phys. Rev. D **36**, 1017 (1987). Their results exceed the scope of this note, but should be kept in mind every time one deals with CS even at a Newtonian level.

[8] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge U. P., New York 1979), p. 221.

[9] Since it is a bounded system, one can simply consider the Tolman mass, which is the integral of ρ_A over the volume of the system that comes out trivially zero in the case of a CS.

[10] R. Gleiser and J. Pullin, "Are cosmic strings gravitationally stable topological defects?" Class. Quan. Grav. **6**, L 141–144 (1989).

A simple proof of the addition theorem for spherical harmonics

Joseph Coster and Harold B. Hart

Department of Physics, Western Illinois University, Macomb, Illinois 61455

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The addition theorem for spherical harmonics is included in many graduate-level texts on mathematical physics, electromagnetism, and quantum mechanics. The theorem says that

$$P_l(\mathbf{n}_1\cdot\mathbf{n}_2)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\mathbf{n}_1)Y_{lm}(\mathbf{n}_2). \qquad (1)$$

Here, $l=0,1,2,\ldots$, and $\mathbf{n}_1$ and $\mathbf{n}_2$ are arbitrary unit vectors whose angular coordinates are (θ_1,ϕ_1) and (θ_2,ϕ_2) so that

$$\mathbf{n}_r=\sin\theta_r\cos\phi_r\mathbf{i}+\sin\theta_r\sin\phi_r\mathbf{j}+\cos\theta_r\mathbf{k} \qquad (r=1,2). \quad (2)$$

Also, $Y_{lm}(\theta_1,\phi_1)\equiv Y_{lm}(\mathbf{n}_1)\equiv Y_{lm}$, and $Y_{lm}(\theta_2,\phi_2)\equiv Y_{lm}(\mathbf{n}_2)\equiv \tilde{Y}_{lm}$. The angle between the two unit vectors is denoted by γ so that $\mathbf{n}_1\cdot\mathbf{n}_2=\cos\gamma$.

Let (θ'_1,ϕ'_1) and (θ'_2,ϕ'_2) be the angular coordinates of the vectors above with respect to a rotated coordinate system where we now call the vectors $\mathbf{n}'_1$ and $\mathbf{n}'_2$. Then $\mathbf{n}_1\cdot\mathbf{n}_2=\cos\gamma=\mathbf{n}'_1\cdot\mathbf{n}'_2$ so that (1) implies that

$$\sum_{m'=-l}^{l}Y^*_{lm'}(\theta_1,\phi_1)Y_{lm'}(\theta_2,\phi_2)=\sum_{m=-l}^{l}Y^*_{lm}(\theta'_1,\phi'_1)Y_{lm}(\theta'_2,\phi'_2). \qquad (3)$$

One may recover (1) from (3) so that (1) and (3) are two equivalent formulations of the addition theorem.

The most frequently found proofs of the addition theorem may be roughly grouped in the three categories below:

(a) Proofs where the addition theorem is a by-product of some specialized formalism such as the theory of angular momentum, group theory, and Green functions. An example of such a proof is the one by Rose.[1] These proofs are often very elegant. Their only drawback is the extensive background required.

(b) Mathematically oriented proofs. The complex-variable proofs of the addition theorem (stated in terms of Legendre polynomials and associated Legendre functions) due to Whittaker and Watson[2] and Copson[3] belong to this category. A related proof is found in Margenau and Murphy.[4] These proofs are rather complicated and lengthy but are often quoted in physics texts.[5-7]

(c) Proofs that use

$$Y_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} c_{mm'} Y_{lm'}(\theta',\phi') \quad (4)$$

[or the special case of (4) where $m = 0$] as a stepping stone to prove (1) and that (in most versions of the proof) base (4) on the properties of a differential equation. Here, (θ,ϕ) and (θ',ϕ') are the angular coordinates of a vector with respect to two rectangular coordinate systems that are rotated with respect to each other. The $c_{mm'}$ are constants that depend on the orientation of the coordinate systems with respect to each other and on l, m, and m'. This type of proof is widely used[8-11] and has a number of attractive features (e.g., it does not require much background on spherical harmonics). However, it is fairly sophisticated and some of the technical details are subtle so that the learner may spend an appreciable time crossing all its t's and dotting all its i's.

Our aim in this note is to present a proof of the addition theorem that is mathematically rigorous but uses simpler mathematics than the proofs above. We prove (1) using induction. The induction procedure is based on some well-known recursion relations for Legendre polynomials and spherical harmonics [Eqs. (7) and (15)]. We do not prove the recursion relations, and we also assume as known Eq. (11) for spherical harmonics and the explicit form for the first few Legendre polynomials and spherical harmonics. Apart from this the proof is self-contained.

To start the induction argument we note that for $l = 0$ Eq. (1) reduces to the identity $1 = 1$.[12]

For $l = 1$ the left-hand side of (1) is $\cos\gamma = \mathbf{n}_1 \cdot \mathbf{n}_2$ which by (2) can be expressed in the form[12]

$$\begin{aligned}\cos\gamma &= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\\ &= a + b + c = (4\pi/3)(Y^*_{11}\tilde{Y}_{11}\\ &\quad + Y^*_{10}\tilde{Y}_{10} + Y^*_{1,-1}\tilde{Y}_{1,-1}),\end{aligned} \quad (5)$$

where

$$a \equiv \tfrac{1}{2}[\exp(-i\phi_1)\sin\theta_1][\exp(i\phi_2)\sin\theta_2], \quad (6a)$$

$$b \equiv \cos\theta_1 \cos\theta_2 = b^*, \quad (6b)$$

and

$$c \equiv \tfrac{1}{2}[\exp(i\phi_1)\sin\theta_1][\exp(-i\phi_2)\sin\theta_2] = a^*. \quad (6c)$$

Having proved (1) for $l = 0$ and 1 we suppose that it is true for $l - 1$ and l, where $l = 1,2,3,\ldots$, and we want to show that it holds for $l + 1$. Accordingly, we can insert (1) on the left-hand side of the recursion relation

$$\begin{aligned}&-lP_{l-1}(\cos\gamma) + (2l+1)\cos\gamma P_l(\cos\gamma)\\ &\quad = (l+1)P_{l+1}(\cos\gamma), \quad (l = 1,2,\ldots)\end{aligned} \quad (7)$$

so that after a division by 4π this equation takes the form

$$\begin{aligned}&-\frac{l}{2l-1}\sum_{m=-l+1}^{l-1} Y^*_{l-1,m}\tilde{Y}_{l-1,m} + \cos\gamma\\ &\times \sum_{m=-l}^{l} Y^*_{lm}\tilde{Y}_{lm} = \left(\frac{(l+1)}{(4\pi)}\right) P_{l+1}(\cos\gamma).\end{aligned} \quad (8)$$

We will prove shortly that

$$\begin{aligned}&-\frac{l}{2l-1}\sum_{m=-l+1}^{l-1} Y^*_{l-1,m}\tilde{Y}_{l-1,m}\\ &+\cos\gamma \sum_{m=-l}^{l} Y^*_{lm}\tilde{Y}_{lm}\\ &= \frac{l+1}{2l+3}\sum_{m=-l-1}^{l+1} Y^*_{l+1,m}\tilde{Y}_{l+1,m} \quad (l = 1,2,\ldots).\end{aligned} \quad (9)$$

Equating the right-hand side of (8) and the right-hand side of (9) completes the induction proof.

We now turn to the proof of (9). We first introduce the notation

$$A_m \equiv aY^*_{lm}\tilde{Y}_{lm}, \quad B_m \equiv bY^*_{lm}\tilde{Y}_{lm}, \quad C_m \equiv cY^*_{lm}\tilde{Y}_{lm}, \quad (m = -l,\ldots,l), \quad (10a)$$

where a, b, and c are defined by (6). By (6b) and (6c)

$$B^*_m = bY^*_{l,-m}\tilde{Y}_{l,-m} = B_{-m} \quad (10b)$$

and

$$C^*_m = aY^*_{l,-m}\tilde{Y}_{l,-m} = A_{-m}, \quad (10c)$$

where we also used

$$Y_{lm}(\theta,\phi) = (-1)^m Y^*_{l,-m}(\theta,\phi) \quad (l = 0,1,2,\ldots; \quad m = -l,\ldots,l). \quad (11)$$

If one inserts (5) into the second term on the left-hand side of (9) and uses the notation (10a), then this term can be expressed in the form

$$\begin{aligned}\cos\gamma \sum_{m=-l}^{l} Y^*_{lm}\tilde{Y}_{lm} &= \sum_{m=-l}^{l}(A_m + B_m + C_m)\\ &= A_l + [A_{l-1} + B_l] + [(A_{l-2} + B_{l-1} + C_l) + (A_{l-3} + B_{l-2} + C_{l-1}) + \ldots\\ &\quad + (A_{-l} + B_{-l+1} + C_{-l+2})] + [B_{-l} + C_{-l+1}] + C_{-l}\\ &= A_l + [A_{l-1} + B_l] + \sum_{m=-l+1}^{l-1}(A_{m-1} + B_m + C_{m+1})\\ &\quad + [B^*_l + A^*_{l-1}] + A^*_l, \quad (l = 1,2,3,\ldots),\end{aligned} \quad (12)$$

where we used (10b) and (10c) in the last step.

We will prove shortly that

$$\begin{aligned}&A_{m-1} + B_m + C_{m+1}\\ &\quad = [(l+1)/(2l+3)]Y^*_{l+1,m}\tilde{Y}_{l+1,m}\\ &\quad + [l/(2l-1)]Y^*_{l-1,m}\tilde{Y}_{l-1,m}\\ &\qquad (m = -l+1,\ldots, \quad l-1),\end{aligned} \quad (13a)$$

$$A_{l-1} + B_l = [(l+1)/(2l+3)]Y^*_{l+1,l}\tilde{Y}_{l+1,l}, \quad (13b)$$

and

$$A_l = [(l+1)/(2l+3]Y^*_{l+1,l+1}\tilde{Y}_{l+1,l+1}. \quad (13c)$$

By (11) it follows from (13b) and (13c) that

$$A^*_{l-1} + B^*_l = [(l+1)/(2l+3)]\, Y^*_{l+1,-l}\, \widetilde{Y}_{l+1,-l} \qquad (14a)$$

and

$$A^*_l = [(l+1)/(2l+3)]\, Y^*_{l+1,-l-1}\, \widetilde{Y}_{l+1,-l-1}. \qquad (14b)$$

Substituting (13) and (14) into the right-hand side of (12) and then substituting (12) into the left-hand side of (9) proves (9). This completes the induction proof except that we still have to prove (13).

To do this we use the recursion relations[13-14]

$$\cos\theta Y_{lm}(\theta,\phi) = [(l-m+1)(l+m+1)]^{1/2} Y_{l+1,m}(\theta,\phi)/U_l + [(l-m)(l+m)]^{1/2} Y_{l-1,m}(\theta,\phi)/V_l \qquad (15a)$$

and

$$\exp(\pm i\phi)\sin\theta Y_{l,m\mp1}(\theta,\phi) = \mp[(l\pm m)(l\pm m+1)]^{1/2} Y_{l+1,m}(\theta,\phi)/U_l \pm [(l\mp m+1)(l\mp m)]^{1/2} Y_{l-1,m}(\theta,\phi)/V_l, \qquad (15b)$$

where $U_l \equiv [(2l+1)(2l+3)]^{1/2}$, $V_l \equiv [(2l-1)\times(2l+1)]^{1/2}$, $l = 1,2,\ldots$, $m = -l+1,\ldots, l-1$, and where in (15b) one keeps either the upper or the lower sign throughout. For $m = l$ these relations are

$$\cos\theta Y_{ll}(\theta,\phi) = Y_{l+1,l}(\theta,\phi)/W_l, \qquad (15c)$$

$$\exp(i\phi)\sin\theta Y_{l,l-1}(\theta,\phi) = -(2l)^{1/2} Y_{l+1,l}(\theta,\phi)/W_l, \qquad (15d)$$

while for $m = l+1$,

$$\exp(i\phi)\sin\theta Y_{l,l}(\theta,\phi) = -(2l+2)^{1/2} Y_{l+1,l+1}(\theta,\phi)/W_l, \qquad (15e)$$

where $W_l \equiv (2l+3)^{1/2}$.

In what follows the complex conjugates of the various Eqs. (15) will also be referred to as Eqs. (15).

Recalling the definitions (6) and (10a) we substitute (15e) into the left-hand side of (13c). This proves (13c). Similarly, one proves (13b) by substituting (15c) and (15d) into the left-hand side. Finally, the substitution of (15a) and (15b) into the left-hand side of (13a) proves (13a) after a simple calculation.

This completes the proof of the addition theorem.

In the above proof we assume that the recursion relations (7) for Legendre polynomials and the recursion relations (15) for spherical harmonics are known. This seems a reasonable assumption since these relations are quite useful,[13] and appear in a large number of texts on mathematical physics and quantum mechanics.

Equation (4) is important in its own right. We now show that (4) is implied by the addition theorem [i.e., by (1) or (3)].

We substitute the expansion ($l = 0,1,2,\ldots$, $m = -l,\ldots,l$)

$$Y_{lm}(\theta'_2,\phi'_2) = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} c_{ll'mm'} Y_{l'm'}(\theta_2,\phi_2), \qquad (16)$$

where the $c_{ll'mm'}$ are constants into the right-hand side of (3) and multiply (3) by $Y^*_{\bar{l}\bar{m}}(\theta_2,\phi_2)$, where $\bar{l} = 0,1,2,\ldots$, $\bar{l} \neq l$, and $\bar{m} = -\bar{l},\ldots,\bar{l}$. Keeping (θ_1,ϕ_1) and (θ'_1,ϕ'_1) fixed we integrate over the entire solid angle Ω_2 where $d\Omega_2 = d\phi_2 \sin\theta_2\, d\theta_2$. By the orthonormality of the spherical harmonics, (3) takes the form

$$0 = \sum_{m=-l}^{l} Y^*_{lm}(\theta'_1,\phi'_1) c_{l\bar{l}m\bar{m}} \qquad (17)$$

so that $c_{l\bar{l}m\bar{m}} = 0$, where $\bar{l} \neq l$ and $\bar{m} = -\bar{l},\ldots,\bar{l}$. Hence we can drop the summation over l' in (16) and set l' equal to l. Then (16) reduces to (4) with $c_{llmm'} \equiv c_{mm'}$.

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[10]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., pp. 100–102.

[11]Reference 7, pp. 169–171. This particular proof of the addition theorem is based on the delta function.

[12]See, e.g., Ref. 8, for tables of the first few Legendre polynomials and spherical harmonics.

[13]See Ref. 8, pp. 699–700.

[14]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), 2nd ed., p. 187.

Comment on "An exotic harmonic oscillator," by J. B. Sztrajman [Am. J. Phys. 58, 159–160 (1990)]

William T. Doyle

Physics Department, Dartmouth College, Hanover, New Hampshire 03755

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In a recent paper in this Journal,[1] Sztrajman has described an interesting and unusual oscillating system consisting of two equal point masses, m, joined by a rigid massless rod of length A and constrained to move in orthogonal directions as shown in Fig. 1(a). Since the rod is inextensible it is clear that the amplitude of oscillation A is fixed, while the frequency of the motion ω may have any value. This contrasts with the usual behavior of a simple harmon-