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Spectral differences between a single gravity shear wave and a continuous superposition of modes

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Abstract: A continuous superposition of gravity waves, propagating upward in a shearing background wind, is analyzed. Starting from the gravity wave equations, it is proven, analytically and numerically, that a uniform spectrum has a resultant Doppler-shifted power spectrum with a −3 power law at large vertical wavenumber contrasting with the −1 power law that characterizes the power spectrum of the monochromatic wave perturbation profile. The preliminary results presented here show that, assuming conservative propagation (no dissipative processes), the Doppler shifting effect can account for observations.

Keywords: Critical level; spectral analysis; Universal spectrum; Doppler shifting.

1. Introduction

Observations of horizontal wind and temperature irregularities exhibit an apparent universal behaviour in their energy distributions at short vertical wavelengths, corresponding to a −3 power law. This spectral shape appears to be independent of the altitude, place, and season (Dewan *et al.*, 1984; Allen and Vincent, 1995) suggesting the presence of physical processes which act on the wave field leading it to a saturated form.

A recent work, Pulido and Caranti (2000), shows that a gravity wave propagating in a linear wind background has a power spectrum (PS) with a −1 slope at large vertical wavenumber provided that the wave amplitude is smaller than the convective saturation amplitude, within the analyzed altitude interval. The shape in the PS is a result of the combined effect of changes in amplitude and vertical wavenumber as functions of height which, in turn, are due to changes in the background wind.

In general, disturbances in the atmosphere are localised and transient. Thus they must be represented by a superposition of a continuum of travelling waves containing a range of values of two independent spectral variables, say ground-based frequency ω and horizontal wavenumber k. Because of background wind shear, the vertical wavenumber, m, cannot be used as a spectral variable since it is not independent of altitude.

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In the literature (Hines, 1991), it is usual to find that the PS resulting from the conservative propagation of a gravity wave spectrum has a -1 power law at large vertical wavenumber. This result is obtained starting from vertical flux conservation, and then averaging in time, while the high resolution observations to be explained correspond to a fixed time (Dewan *et al.*, 1984; Allen, 1995). As the obtained -1 power law is not in agreement with observed power spectra, it has been suggested that a dissipative process must be determining the amplitudes at the large wavenumber end of the spectral tail.

In this paper, the vertical profile (at a fixed time) corresponding to a continuous spectrum of travelling shear waves is obtained; the asymptotic behaviour of the vertical wavenumber PS is then calculated theoretically from Parseval's theorem. For wavenumbers greater than the largest incident wavenumber a spectral tail is produced by Doppler shifting, and the corresponding asymptotic behaviour is achieved. The theoretical prediction, as well as the numerical PS estimation, show that this spectral tail has a -3 power law.

2. The general solution

We consider inviscid gravity wave perturbations in mean flow conditions which are given by a positive, uniform buoyancy frequency ($N^2 > 0$) and a horizontal wind $U = U(z)$. It is assumed that the frequency of the waves is much higher than the inertial frequency. Under the Boussinesq approximation, a single equation can be obtained for the vertical wind perturbation (for an introduction to the treatment of gravity waves see Lighthill, 1978). As we are interested in perturbations of vertical scales smaller than the horizontal ones, the linearised gravity wave equation results

$$(\partial_t + U(z)\partial_x)^2 \partial^2_{zz} w + N^2 \partial^2_{xx} w = 0 \tag{1}$$

Assuming the background wind linear, it is possible to find an exact analytical solution for this equation. The reference level is chosen at $z = 0$, where the background wind is zero ($U(z) = \mathrm{d}_z U\, z$). Proposing a solution of the form $w = \widehat{w}(z) \exp i(\omega t - kx)$ and replacing in eqn (1), the following single equation in z is obtained

$$\mathrm{d}^2_{zz}\widehat{w} + \frac{N^2 k^2}{(\omega - \mathrm{d}_z U k z)^2}\widehat{w} = 0 \tag{2}$$

The solution of this second order equation is

$$\widehat{w} = w_1\left(1 - \frac{\mathrm{d}_z U k}{\omega} z\right)^{1/2 - i\mu} + w_2\left(1 - \frac{\mathrm{d}_z U k}{\omega} z\right)^{1/2 + i\mu} \tag{3}$$

where $\mu = (R_i - 1/4)^{1/2}$ ($R_i = \frac{N^2}{\mathrm{d}_z U^2}$ is the Richardson number), w_1 (w_2) is the amplitude of the wave propagating upward (downward). In general, mean atmospheric flows satisfy $R_i \gg 1$ therefore it can be assumed that $\mu \approx R_i^{1/2}$. This work concentrates on gravity waves which are propagating upward and approaching a critical level at $z_c = \frac{\omega}{\mathrm{d}_z U k}$; therefore we take $\mathrm{d}_z U > 0$ and $w_2 = 0$.

For an arbitrary background wind geometry, eqn (3) represents the solution for the first order of the background wind power series expansion in the vicinity of z_c given by the Frobenius method (e.g. Booker and Bretherton, 1967). The first order is dominant at large vertical wavenumber (near the critical level z_c). Therefore, the spectral asymptotic behaviour to be obtained may be applied to any background wind provided that $d_z U = d_z U(z_c) \neq 0$.

Using the continuity equation ($\partial_x u + \partial_z w = 0$), the vertical structure of the horizontal wind perturbation of a wave propagating upward is,

$$\widehat{u}(z) = u_1 \left(1 - \frac{d_z U k}{\omega} z\right)^{-1/2 - i\mu} \tag{4}$$

Figure 1 shows the behaviour of eqn (4). As the wave approaches a critical level, amplitude and vertical wavenumber become unbounded. Thus, dissipative processes must be considered between the overturning condition ($N^2_{mean} + N^2_{wave} = 0$) and the critical level

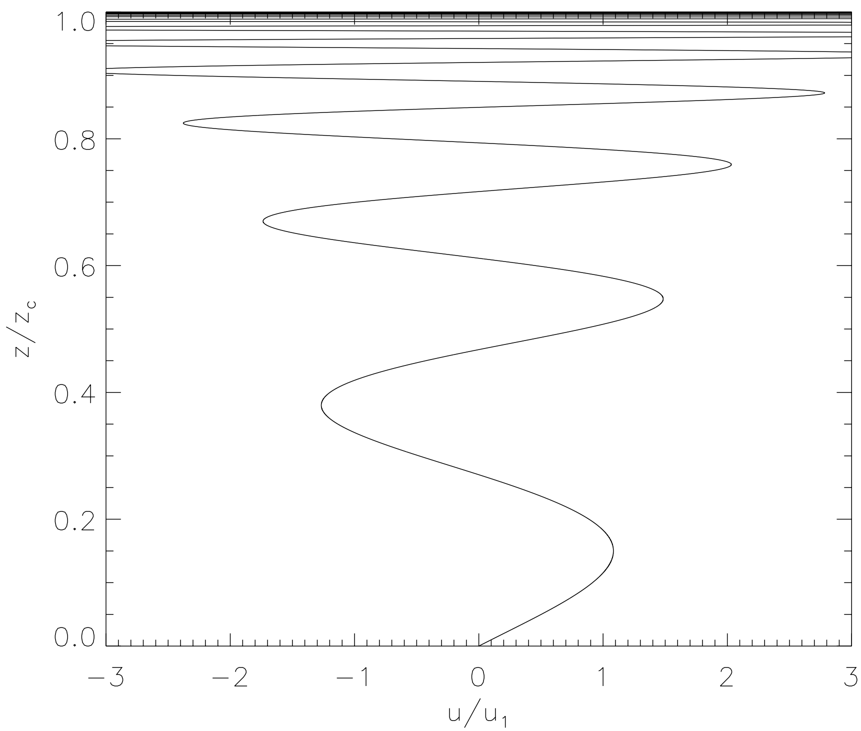

Figure 1. Horizontal velocity perturbation corresponding to a single gravity shear wave (Imaginary part of eqn (4)). Altitudes are in units of the critical level height, z_c, and amplitudes are in units of the amplitude at $z = 0$. The only free parameter, the Richardson number, is taken to be $R_i = 100$.

(Chimonas, 1997). Here, the main objective is to compare the spectral features of a continuous superposition of waves with those of a single wave under conservative propagation. Therefore, the solution of eqn (4) in the neighbourhood of the critical level is just considered for comparison with the continuous superposition profile.

General disturbances can be represented by a linear combination of different modes. In order to specify a general disturbance in the spectral domain we have to choose two linearly independent variables in the spectral space; the third variable is given by the dispersion relation (the modes in the spectral space are on a surface that satisfies the dispersion relation). As our interest is to see how Doppler shifting changes the vertical PS, we express the solution as a continuous superposition of modes, each of which is a function of the incident vertical wavenumber m_0 (which is the vertical wavenumber at the reference level $z = 0$), and the horizontal wavenumber k,

$$w(x, z, t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}m_0 \int_0^\infty \mathrm{d}k\widetilde{w}(k, m_0) \cdot (1 - m_0 R_i^{-1/2} z)^{1/2 - i\mu} \exp i(\omega t - kx) \tag{5}$$

It can be readily proven that the expression of eqn (5) is a solution of eqn (1) when the source spectrum satisfies its dispersion relation, namely, $\omega = \frac{Nk}{m_0}$.

The horizontal perturbation corresponding to the disturbance eqn (5) is

$$u(x, z, t) = \frac{1}{\pi}\int_0^\infty \int_0^\infty \tilde{u}(k, m_0)(1 - m_0 R_i^{-1/2} z)^{-1/2 - i\mu} \cdot \exp i(\omega t - kx)\, \mathrm{d}m_0 \mathrm{d}k \tag{6}$$

where $\tilde{u}(k, m_0) = \frac{m_0}{k}\tilde{w}(k, m_0)$, $\tilde{u}(k, m_0)$ represents the spectral amplitude of the horizontal wind perturbation in a zero wind background. It may be interpreted that the disturbance is generated at $z = 0$ and so this source spectrum will be the "incident" spectrum in the interest region, $z > 0$. Note also that the incident spectrum is free of Doppler effects.

For the present analysis we consider disturbances that belong to a broad incident spectrum which has an arbitrary horizontal wavenumber spectral shape and is uniform in incident vertical wavenumber (white noise). The general case for arbitrary vertical wavenumber shapes is analyzed in a companion paper. There, it is shown that the results presented here hold in the general case. The spectral amplitude considered has the form:

$$\widetilde{u}(m_0, k) = \begin{cases} \pi A S(k) & m_{01} \le m_0 \le m_{02} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In order to examine the vertical wavenumber PS we will calculate the profile of this superposition of waves at a fixed time and horizontal position (e.g. $u_p(z) = u(x = 0, z, t = 0)$). Replacing eqn (7) in eqn (6) an analytical expression can be obtained for the wave perturbation profile,

$$u_p(z) = -\frac{u_a R_i^{1/2}}{(1/2 - i\mu)}\frac{1}{z}\left[(1 - m_{02} R_i^{-1/2} z)^{1/2 - i\mu} - (1 - m_{01} R_i^{-1/2} z)^{1/2 - i\mu}\right] \tag{8}$$

where $u_a = A\int S(k)\mathrm{d}k$. The wave amplitude at $z = 0$ is $u(z \to 0) = u_a(m_{02} - m_{01})$. For the considered disturbance, because of the dispersion relation a range of values of m_0 corresponds to a range of values for the phase velocity $c_x = \frac{N}{m_0}$. Then, there is a different critical level for each component between $z = R_i^{1/2} m_{02}^{-1}$ and $z = R_i^{1/2} m_{01}^{-1}$. The profile eqn (8) has features which are qualitatively different from eqn (4); when the continuous superposition approaches the first critical level ($z \longrightarrow z_c$) the amplitude of the oscillations goes to 0 as $\frac{(1-z/z_c)^{1/2}}{z}$ (as seen in Figure 2). In the case of a single gravity wave the amplitude is $(1 - z/z_c)^{-1/2}$ which is unbounded as $z \longrightarrow z_c$.

The singularity for each separate component is not present in the complete velocity perturbation and the linearized equations remain valid for the vertical scales that we are interested in. The convective instability produced by the high shear of the perturbation in the vicinity of the critical level is entirely negligible (taking $R_i = 100$, $m_{02} = \frac{2\pi}{5\text{ km}}$ and a disturbance amplitude of $u(z=0) = 1$ m/s, this instability will occur within a 1.2 m layer with amplitudes smaller than 0.0015 m/s and vertical wavelengths shorter than 1 m).

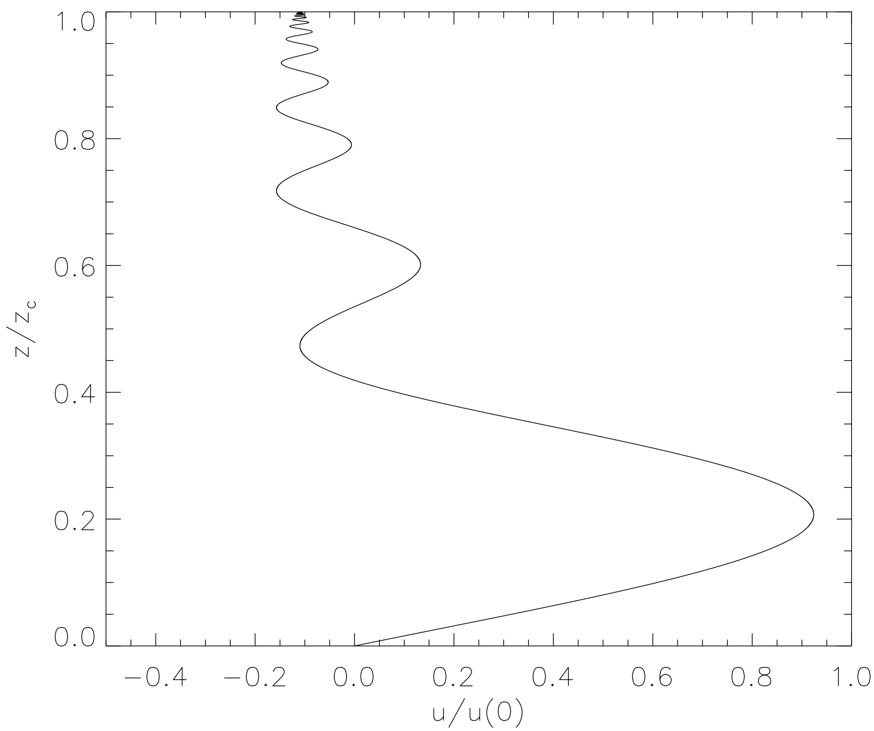

Figure 2. Horizontal velocity perturbation profile corresponding to a uniform incident spectrum of gravity shear waves (Imaginary part of eqn (8)). Units as in Figure 1. The value of parameters are $R_i = 100$ and $m_{01} = 0.25 m_{02}$.

3. The Doppler shifted spectrum

The PS of a single gravity wave propagating in a background wind shear has been calculated at large vertical wavenumber by Pulido and Caranti (2000). For a profile of the form eqn (4) the PS is,

$$PS(m) = \frac{|u_1|^2 R_i^{1/2}}{L m_{01}} \frac{1}{m} \tag{9}$$

where L is the length of the analyzed altitude interval and $m_{01} = \frac{Nk}{\omega}$ is the fixed incident wavenumber. Figure 3 shows the PS estimation and the theoretically predicted one corresponding to the wave perturbation profile shown in Figure 1.

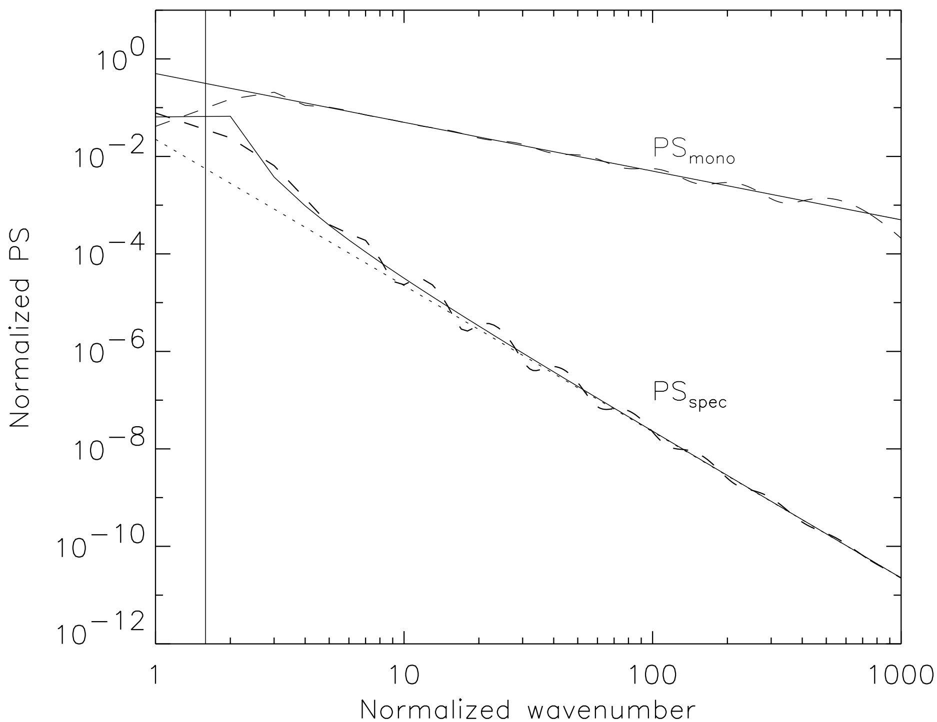

Figure 3. Power spectrum estimation PS_{mono} corresponding to the single gravity wave of Figure 1 and the theoretical prediction eqn (9). Power spectrum estimation PS_{spec} corresponding to the profile of Figure 2 calculated using the prewhitening and postdarkening method (dashed line) and the theoretically predicted PS, eqn (11), (continuous line). The vertical line indicates the highest wavenumber m_{02} present in the incident spectrum. Dotted line is the -3 asymptotic behavior eqn (12).

In order to find the asymptotic behaviour of the PS for a continuous spectrum of travelling waves, the total power of the profile eqn (8) is calculated. When the analyzed altitude interval is between $z=0$ and the first critical level z_c (Figure 2), the asymptotic behaviour at high vertical wavenumber is dominated by the first oscillatory term in eqn (8). Therefore, the remaining oscillatory term and interference terms can be neglected for $m \gg m_{02}$. Thus, the total power yields

$$\int u_p(z) u_p(z)^* \mathrm{d}z = |u_a|^2 \int \frac{1}{z^2} (1 - m_{02} R_i^{-1/2} z) \mathrm{d}z \tag{10}$$

Following a mathematical procedure similar to that of Sato and Yamada (1994), a variable change is performed in eqn (10) from z to $m = \frac{m_{02}}{1 - m_{02} R_i^{-1/2} z}$. Then, using the Parseval theorem ($\int PS \, \mathrm{d}m = L^{-1} \int |u_p|^2 \, \mathrm{d}z$) the PS is given by

$$PS = \frac{|u_a|^2 R_i^{-1/2}}{L} \frac{m_{02}^3}{m^3} \left(1 - \frac{m_{02}}{m}\right)^{-2} \tag{11}$$

for $m > m_{02}$.

The upper extreme of the interval at z_c ensures the existence of large vertical wavenumbers. From a spectral point of view, it is necessary for the incident spectrum to extend up to $m_{02} \geq \frac{N}{U_{max}}$ in order to generate the whole spectral tail.

The asymptotic behaviour of eqn (11) is,

$$PS = \frac{|u_a|^2 R_i^{-1/2}}{L} \frac{m_{02}^3}{m^3} \tag{12}$$

The spectral tail has a -3 slope for vertical wavenumbers greater than the maximum incident wavenumber. Another similar contribution, proportional to m_{01}^3 rather than to m_{02}^3, would be obtained if all the critical levels were included in the interval where the Fourier transformation is carried out.

The prewhitening and postdarkening method, which is examined in Dewan and Grossbard (2000), is used in the numerical calculation of the Fourier transform of the profile (8) in order to diminish leakage effects. It should be noted that it is not possible to apply windowing because the large wavenumber portion of the profile is located in the upper extreme. Figure 3 shows the theoretical prediction at large wavenumber (eqn (11)) and the numerically calculated PS. The -3 asymptotic behaviour is soon achieved for $m > m_{02}$. Figure 3 clearly shows the difference between the behaviour in the spectral tail of a single wave and of a continuous superposition of them.

4. Discussion

There are differences in the resulting PS of wind perturbation corresponding to single gravity shear waves and a continuum superposition of these modes. The interference between the different components of the spectrum produces a different behaviour at large wavenumber. While each component (single wave) has a -1 spectral tail, a continuous superposition of them presents an asymptotic shape which matches a -3 power law in the particular case analyzed.

The Doppler spread theory proposed by Hines (1991) was based on the conservation of vertical momentum flux. As it has been illustrated here, the results cannot be extrapolated for every fixed time to continuous spectra of waves, for which a different behaviour is obtained. This feature indicates that the general disturbance of Figure 2 does not conserve vertical flux at a fixed time because it does not have a fixed frequency (Lighthill, 1978), and also that the PS resulting from a continuous superposition of gravity shear waves is different from the sum of the power spectra of each mode since these modes are not orthogonal in the m, k spectral space.

This counterintuitive result can be traced back to the vertical structure of a single mode which has an amplitude and a vertical wavenumber dependent on z. Thus, modes may have destructive interference at large m wavenumber (note that the modes are defined in the m_0, k spectral space but not in the m, k space) as it happens in the analyzed case. These interference effects lead to a spectral power law for a profile different from the one resulting from the time averaged power spectrum.

I find it sensible to believe that the problem found here is related to the interpretation of profiles and power spectra derived from measurements. That is, in a uniform frame (without background wind shear), one could be led to think that the profile of Figure 2 and its PS (see Figure 3) belong to a wave, or a group of waves, that is suffering dissipative processes (Dewan and Good, 1986; Weinstock, 1990). These preliminary results indicate that it is adequate to parameterise the gravity wave momentum deposition taking into account only the critical layer absorption of the waves.

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