GEOPHYSICAL RESEARCH LETTERS, VOL. 27, NO. 1, PAGES 101-104, JANUARY 1, 2000

Power spectrum of a gravity wave propagating in a shearing background wind

Manuel Pulido and Giorgio Caranti

FaMAF, Universidad Nacional de Cordoba, Argentina

Abstract. We re-analyze the effect on the spectral tail of a gravity wave propagating in a shearing background. The contribution to the spectrum of horizontal wind perturbations for low vertical wavenumbers comes from the Doppler shifting with a -1 slope and for high wavenumbers from the leakage effect with slopes ranging between -4 to -2. If leakage is not present, it becomes necessary to consider the termination of the wave in order to account for the spectral tail. We show that the decreasing of the wave amplitude when the overturning starts, leads to a spectral slope of -3, and an amplitude proportional to N^2 (the square of the Brünt Väisälä frequency).

Introduction

The energy distribution of horizontal winds and temperature irregularities has an apparent universal behavior in short vertical wavelength this behavior was first noted in measured profiles by *VanZandt* [1982]. There are several works that associate this fact with different physical interpretations, [*Dewan and Good*, 1986; *Weinstock*, 1990; *Hines*, 1991a].

In the last few years, there are works showing that the existence of the spectral tail is correlated with the wave termination. *Sato and Yamada* [1994] showed that the spectrum of one terminating wave with saturated amplitude has a -4 slope for internal waves while for inertial waves the slope is -3. *Chimonas* [1997] modeled the wind irregularities with a train of gravity waves which have random propagation speed. He studied two cases; in the first one the waves terminate abruptly in the overturning condition (where the Brünt Väisälä frequency is equal to 0) which leads to spectral slopes in a range between -3.1 to -2.4, in the second one the waves have an oscillating decay between the overturning condition and the critical level generating spectral tails lying in the range from -3.9 to -3.6. *Pulido and Caranti* [1999] presented evidences in a measured profile that the contributions to the spectral tail come from localized heights. This fact suggested the presence of sporadic wave termination rather than a saturated spectral tail [*Smith et al.*, 1987].

Recently, *Giraldez and de la Torre* [1998] (hereinafter GdelaT) analyzed the spectrum of a wave that is refracted by a linear background wind. Their conclusion was that the slope is -3.

We re-analyze this case and our conclusions are different. We found that the spectral tail in the profiles as those shown by GdelaT have two different contributions. For wavenumber lower than the highest wavenumber present in the profile, a -1.5 slope is produced by Doppler shifting. Here we understand "the wavenumber present in the profile" of this single wave at each altitude as the parameter given by the dispersion relation. In higher wavenumber the spectral tail is generated by the discontinuity at the extremes of the interval. In section 2 we formalize these results and in the following one we use the correct wave function for a linear background wind and study the effect of the wave termination.

Paper number 1999 1999GL010819 .
0094 8276/99/1999 1999GL010819$05.00

Comments on the work by Giraldez and de la Torre

In an attempt to explain the spectral tail of horizontal wind perturbations of vertical profiles GdelaT pointed towards the wind shear Doppler shifting as a possible cause. In order to do so they considered a single wave propagating in a stratified medium with a constant shear. The wave function they chose to simulate the wave propagation was the following:

$$u(z) = u(z_0)\ \exp(i\ k(z)\ z) \tag{1}$$

where $k(z) = \frac{k_0}{1-U(z)k_0/N}$, N is the Brunt Vaisala frequency, k_0 is the initial vertical wavenumber (equal to N/c, c phase speed) and $U(z)$ is the horizontal background wind.

We would like to point out that the function used to simulate the wave does not satisfy the wave equation. In fact, the vertical velocity wave equation is given by, [*Gill*, 1982],

$$\frac{d^2w}{dz^2} + k(z)^2 w = 0, \tag{2}$$

and using the continuity equation that gives a relation between u and w a similar wave equation is obtained for u. By inspection one can verify that (1) is not a solution.

The approximation is correct when the shear background can be neglected, but in this case the wave would not be refracted and the wavenumber remains constant.

On the other hand, if the function for the wave is considered correct, the power spectrum has an analytical solution (see the appendix I), in particular when the background wind is linear ($U = d_zU\ z$), the power spectrum density (PSD) will be

$$PSD(k) = \frac{u(z_0)^2 k_0^{1/2}}{2\ A\ L\ k^{3/2}} \tag{3}$$

where $A = \frac{d_zU\ k_0}{N}$, and L is the length of the interval. This expression is only valid for a range of vertical wavenumbers, $k_0 < k < k_1$, where the extremes of the interval are smallest and largest value of $k(z)$ in the wave equation.

The PSD has a slope of -1.5 in the above wavenumber range.

For $k > k_1$ the asyntotic expansion given by *Pulido and Caranti* [1999] can be used, giving a Fourier coefficient for $u(z)$,

$$C(k) = \frac{1}{\sqrt{L}} \sum_{s=0}^{\infty} \frac{1}{(ik)^{s+1}} (u^{(s)}(z_1) - u^{(s)}(z_0)) \quad (4)$$

where $L = z_1 - z_0$, and $u^{(s)}(z)$ is the s-derivative of the real function $u(z)$.

Taking into account only the first two term of the series, the PSD, which is $|C(k)|^2$, will be given by

$$PSD(k) = \frac{1}{L}[\frac{1}{k^2}(u(z_1) - u(z_0))^2 + \frac{1}{k^4}(u^{(1)}(z_1) - u^{(1)}(z_0))^2] \quad (5)$$

One term contributes with a k^{-2} and the other with k^{-4} and since the latter, being the derivative of a wave function, is out of phase with the former, the leakage can not be avoided. Leakage being a sort of energy transfer from low to high wavenumbers (see *Pulido and Caranti* [1999])

Let us analyze the results presented in GdelaT. They selected specific parameters for (1): $U = 10^{-4} z$ s^{-1} and $\lambda_0 = \frac{2\pi}{k_0} = 3$ km. In Figure 1 the full line shows the profile for these conditions; the Doppler shifting goes from $k(0) = k_0$ to $k(L) = 1.6\ k_0$. The spectrum is shown in Figure 2. The region between vertical lines represents the region of the refraction (only three Fourier harmonics) while the spectral tail is completely generated by leakage. The straight line shows the leakage produced by the discontinuity at the extremes ($\frac{1}{Lk^2}(u(L) - u(0))^2$).

The other cases considered in GdelaT only represent changes in the leakage effect. That is, the changes in the spectral tail are not due to the parameters used by them. In fact they noted that the slopes range between -2 to -4 as it can be seen from (5). For example, if the parameters are $U = 10^{-4} z^{1.05}$ and $\lambda_0 = 3$km, the profile does not have first order discontinuity at the extremes, as it is shown in Figure 1 (dashed line). Therefore, the tail will be generated by the discontinuity in the derivative and following (5) the slope will be -4. Figure 3 shows the power spectrum and the approximation given by (5); again we have a negligible Doppler shifting and the spectral tail is only related to the leakage effect.

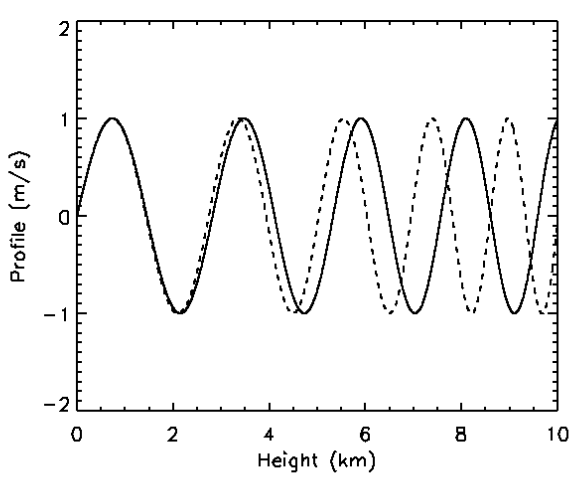

Figure 1. Profile (1), with continuous line $\lambda_0 = 3$ km, $U = 10^{-4} z$ 1/s, with dashed line $\lambda_0 = 3$ km, $U = 10^{-4} z^{1.05}$

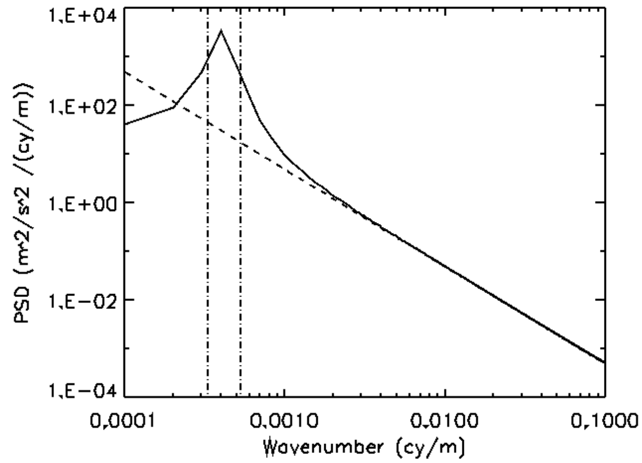

Figure 2. Power spectrum for the profile with continuous line of Figure 1, the straight line represents the first order in the expansion (5). Vertical lines show the lowest and the highest wavenumber present in the profile

On the PSD when the wave termination is considered

The solution to the wave equation (e.g. eq. (2)) when a linear wind background is present, is

$$w(z) = w_0\ (1 - A\ z)^{1/2 - i\ \mu} \quad (6)$$

where $\mu = \sqrt{N^2/d_z U^2 - 1/4}$, w represents the vertical perturbation velocity, and w_0 is the vertical perturbation at the reference level, $z = 0$; the horizontal velocity can be obtained from the continuity equation

$$u(z) = u_0\ (1 - A\ z)^{-1/2 - i\ \mu} \quad (7)$$

where u_0 is the amplitude of the perturbation at $z = 0$. Replacing in (A5) the power spectrum will be

$$PSD(k) = \frac{u_0^2}{LA\ k} \quad (8)$$

The interval may contain a critical level, that is, the wind background equals to the phase speed $z_c = 1/A$. Near a critical level the wave will become unstable and will start to loose energy. The static instability condition is $N_T^2 = 0$ where N_T is the Brunt Vaisala frequency evaluated with the mean plus, the potential temperature perturbation [*Hodges*, 1967]. The resulting threshold velocity will be given by:

$$u(z_b) = \frac{1/2 - i\mu}{-1/2 - i\mu}\ \frac{N}{k(z_b)} \quad (9)$$

It should be noticed that the above overturning condition is applicable for a quasi monochromatic linear wave, that is, when there are not any non linear wave effects [*Hines*, 1991a].

When the wave has a saturation amplitude, say $u(z_b) = N/k(z_b)$ since $\mu \gg 1$, the overturning starts. We suppose

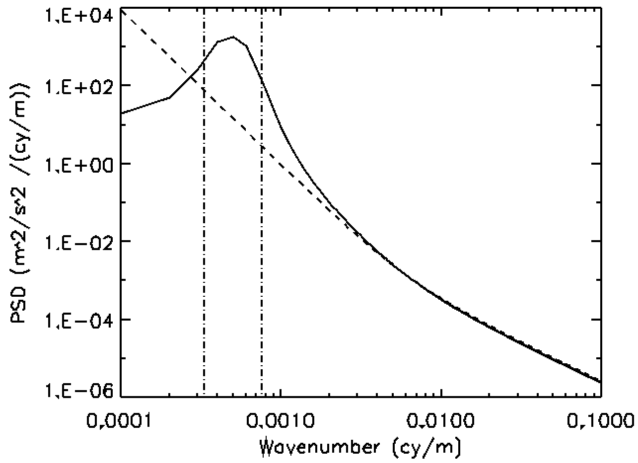

Figure 3. PSD for the profile with dashed line of Figure 1. Dashed line represents the expansion (5). Vertical lines represent the lowest and the highest wavenumber present in the profile

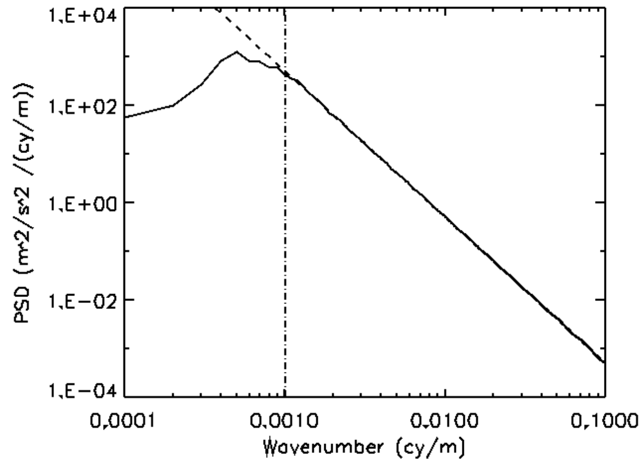

Figure 5. PSD for the profile of Figure 4. Dashed line represents the theoretical prediction (11). Vertical line is the wavenumber where the overturning begins

that the wave terminates at the critical level and the perturbation profile between overturning condition and the critical level can be expressed as [*Fritts*, 1985; *Chimonas*, 1997],

$$u(z) = \frac{u_0}{1 - A\ z_b}\ (1 - A\ z)^{1/2 - i\ \mu} \tag{10}$$

The contribution to the PSD will be (see Appendix I and II),

$$PSD(k) = \frac{u(z_b)^2 \mu^2 A}{L k^3} \tag{11}$$

This is an important result since not only the -3 slope is recovered but it is also proportional to N^2 through the μ dependencies (note that A and $u(z_b)$ are independent of N); which predicts the changes in the power spectra amplitude between the troposphere and the stratosphere [*Dewan and Good*, 1986].

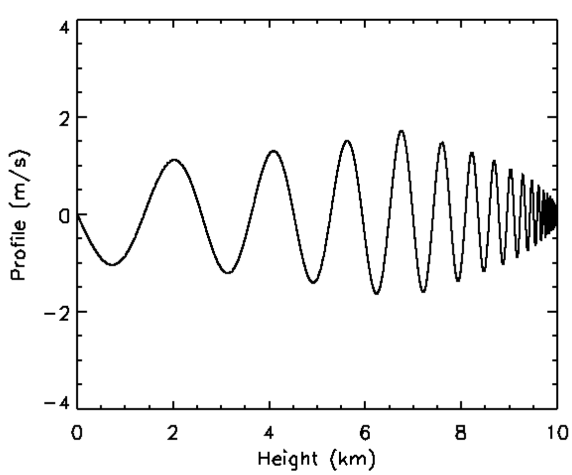

Figure 4. Gravity wave propagating in a linear wind, with $\lambda_0 = 3$ km and $u = 1$ m/s and $N = 0.01$ 1/s. When the wave is saturated the profile is given by (10)

Figure 4 shows the profile of a wave that entered the saturation and fades away as it approaches the critical level while increasing the spatial frequency rapidly. The parameters used for this wave were $\lambda_0 = 3$km, $u_0 = 1$m/s and $A = 1/10km^{-1}$ since the critical level is located at the end of the interval.

The power spectrum is shown in Figure 5, and it is superimposed with the theoretical prediction which is valid for $k > k(z_b)$, and for the parameters of figure 4, $k(z_b) = 0.001$ cy/m.

Conclusions

We have reinterpreted the Doppler shifting effect on one wave by a constant background shear. Our results show that the spectral slope is -1, as long as there is no critical level within the height interval.

The effect shown in *Giraldez and de la Torre* [1998] is entirely due to leakage. That is, the amplitudes in large wavenumbers, for those analyzed cases, are caused by the discontinuities at the extremes of the interval. The changes in the wavenumber due to refraction are not important.

If one accepts the assumption that a wave propagating analysis can be extrapolated to measured spectra, it is necessary to consider the wave termination to account for the broad spectrum when leakage is not present. When a wave is terminating, the combined effects of decaying amplitude and Doppler shifting result in power spectra, with -3 slopes, and amplitude proportional to N^2.

It is quite interesting to see how the Doppler shifting leads to spectral shapes according to the functional form of the background wind (A5). This shows the way to deal with the problem of evaluating the effects a wave field exerts upon one particular wave while the WKB approximation is still valid. However, the corresponding analysis is out of the scope of the present work.

Finally, there are several effects that should be kept in mind to interpret the small scale of the wind irregularities. Here we point out the Doppler shifting and the wave termination not as the only cause for the observations but as important factors contributing to them.

Appendix I

Suppose that the horizontal wind perturbation has a general form,

$$u(z) = g(z)exp(if(z)), \tag{12}$$

where $g(z)$ and $f(z)$ are known functions through the WKB approximation for a background wind $U(z)$, the exact solution to the linear wind case, or (1).

Taking into account the Parseval theorem,

$$\int PSD(k)dk = \frac{1}{L}\int_{z_0}^{z_1} |u(z)|^2 dz \tag{13}$$

then replacing u(z),

$$\int PSD(k)dk = \frac{1}{L}\int_{z_0}^{z_1} |g(z)|^2 dz \tag{14}$$

When the amplitude, $g(z)$ varies slowly with z, the wavenumber is given by $k(z) = f'(z)$, where primes represent the derivative with respect to z. If $f'(z)$ is a monotonic function, changing the variable from z to k in the integral using the inverse function, $z = f'^{-1}(k)$,

$$\int PSD(k)dk = \frac{1}{L}\int_{k_0}^{k_1} \frac{|g(f'^{-1}(k))|^2}{f''(f'^{-1}(k))}dk, \tag{15}$$

where $k_1 = \max\{k(z_0), k(z_1)\}$, $k_0 = \min\{k(z_0), k(z_1)\}$.

Since the equality holds while the extremes of the integration interval are free, the integrands must be equal, that is

$$PSD(k) = \frac{|g(f'^{-1}(k))|^2}{L\ f''(f'^{-1}(k))} \tag{16}$$

The above procedure can be considered as a generalization of a similar one by *Sato and Yamada* [1994]. (A5) is a more general equation describing the effect of the Doppler shifting of a wave on its spectrum. In particular, this is so for any background wind abiding the WKB approximation.

Appendix II

An alternative proof of the PSD for a wave propagating in a linear background wind ((7) and (10)), is as follows; by definition, the Fourier coefficient for $u(z)$ is,

$$C(k) = \frac{1}{\sqrt{L}}\int_{-\infty}^{\infty} H(1-A\ z)\ (1-A\ z)^{\alpha - i\mu} exp(-i\ k\ z)dz \tag{17}$$

where $H(1-Az)$ is the Heaviside function. Changing variable from z to $z' = k(1-A\ z)$ we obtain,

$$C(k) = \frac{1}{\sqrt{L}}\frac{e^{-ik/A}}{A}k^{-(1+\alpha)+i\ \mu}\int_{-\infty}^{0} z'^{\alpha - i\mu} exp(i\ z'/A)dz' \tag{18}$$

where the independent of k integral is only function of A, μ and α. Multiplying $C(k)$ by the complex conjugate the dependencies of k in (8) and (11) are recovered.

Acknowledgments. This work was supported by CONICET - Argentina.

References

Chimonas, G., Waves and the middle atmosphere wind irregularities. *J. Atmos. Sci., 54*, 2115-2128, 1997.

Dewan, E. M. and Good R. E., Saturation and the "universal" spectrum for vertical profiles of horizontal scalar winds in the atmosphere. *J. Geophys. Res., 91*, 2742-2748, 1986.

Fritts, D. C., Gravity wave saturation in the middle atmosphere: A review of theory and observations. *Rev. Geophys., 22*, 275-308, 1985.

Gill, A. E., *Atmosphere-Ocean dynamics*, 662 pp., Academic Press, 1982.

Giraldez A. and A. de la Torre, On the effect of the mean shear wind in the predicted "universal" spectral shape of the atmospheric variables. *Geophys. Res. Lett., 25* , 3521-3524, 1998.

Hines C. O., The saturation of gravity waves in the middle atmosphere. Part I: Critique of linear-instability theory. *J. Atmos. Sci., 48*, 1348-1359, 1991a.

Hines C. O., The saturation of gravity waves in the middle atmosphere. Part II: Development of Doppler-spread theory. *J. Atmos. Sci., 48*, 1360-1379, 1991b.

Hodges R., Jr., Generation of turbulence in the upper atmosphere by internal waves. *J. Geophys. Res., 72*, 3455-3458, 1967.

Pulido M. and G. Caranti, Spectral tail of a gravity waves train propagating in a shearing background. Submitted to *J. Atmos. Sci.*, 1999.

Sato K. and M. Yamada, Vertical structure of atmospheric gravity waves revealed by the wavelet analysis. *J. Geophys. Res., 99* , 20623-20631, 1994.

Smith S. A., D. C. Fritts and T. E. VanZandt, Evidence for a satured spectrum of atmospheric gravity waves. *J. Atmos. Sci., 44*, 1404-1410, 1987.

VanZandt, T. E., A universal spectrum of buoyancy waves in the atmosphere, *Geophys. Res. Lett., 9*, 575-578, 1982.

Weinstock, J., Satured and unsaturated spectra of gravity waves and scale-dependent diffusion. *J. Atmos. Sci., 47*, 2211-2225, 1990.

M. Pulido and G. Caranti Facultad de Matematica Astronomia y Fisica Universidad Nacional de Cordoba Ciudad Universitaria (5000) Cordoba, Argentina (e-mail: pulido@roble.fis.uncor.edu)

(Received May 12, 1999; revised July 14, 1999; accepted August 2, 1999.)